Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

al proposition. For the right hand branch of the hyperbola (9) in III, IV, and V crosses the circle at $x=+\frac{1}{2}a$ and is inside from $x=+\frac{1}{2}a$ to $x=+\frac{1}{2}d$. The left hand branch begins at $x=-\frac{1}{2}d$ and leaves the circle at $x=-\frac{1}{2}a$. Between the limits suggested by equating the radicals to zero ($x=+\frac{1}{2}a$ and $x=-\frac{1}{2}a$) there are two pieces of curve, one of which contains points filling the given conditions, the other containing points violating them. That the tentative methods adopted by me are wholly unsatisfactory I freely admit, and yet I feel sure that graphical methods will be of great aid in establishing a theory of irrational equations.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

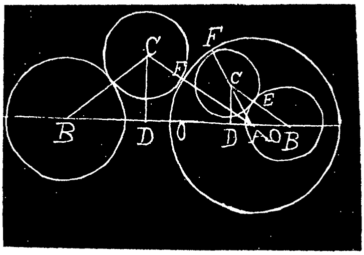
Let $AF=R$, $BE=r$, $CF=CE=t$.

Then when B is inside of circle A we have $BC+AC=r+t+R-t=R+r$.

$\therefore$ The locus of C, the point equidistant from both circumferences, is an ellipse, foci A, B.

When B is without A we have $AC-BC=R+t-r-t=R-r$.

$\therefore$ The locus of C is a hyperbola, foci A, B. Also as follows:



Let the mid-point O, between A and B be origin, $AB=2a$, $OD=x$, $CD=y$.

$$\therefore\ BC^2=(r+t)^2=(a\pm x)^2+y^2 \ldots.(1).$$

$$AC^2=(R\mp t)^2=(x\mp a)^2+y^2 \ldots(2).$$

Eliminating t between (1) and (2) we get

$$r\pm R\mp\sqrt{[(x\mp a)^2+y^2]}=\sqrt{[(x\pm a)^2+y^2]}, \text{ or } \frac{4x^2}{(R\pm r)^2}+\frac{4y^2}{(R\pm r)^2-4a^2}=1.$$

This equation shows that when one circle is within the other $(R+r)>2a$ and the locus is an ellipse with semi-axes $\frac{1}{2}(R+r)$ and $\frac{1}{2}\sqrt{[(R+r)^2-4a^2]}$.

When circles are external the locus is an hyperbola with semi-axes $\frac{1}{2}(R-r)$ and $\frac{1}{2}\sqrt{[4a^2-(R-r)^2]}$.

If the circles intersect, for internal contact to one circle and external to the other we have the same ellipse; for both internal and both external we have the same hyperbola as above.

Also solved by *GUY B. COLLIER, CHARLES C. CROSS, WALTER H. DRANE, ALOIS F. KOVARIK, W. W. LANDIS, J. SHEFFER*, and *COOPER D. SCHMITT*.

DIOPHANTINE ANALYSIS.

73. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Find integral values for x and y in $\begin{pmatrix} 2x^2-y^2=\square \\ 2y^2-x^2=\square \end{pmatrix}$.

Discussion by W. T. KING, Ottawa, Canada.

We may suppose x and y to have no common factor ; for if they have one it will also be a common factor of a and b, and its square may be divided out of both equations, leaving equations of the same form. For the same reason no two of x, y, a, or of x, y, b can have a common factor, nor can a and b, without its being common to all. We therefore suppose that x, y, a and b are all prime to one another. Also we may assume without loss of generality that $x>y$. Then it is evident that $a>x$, $b<y$.

From the first equation we have

$$2(a^2-x^2)=a^2-y^2.$$

Suppose $\dfrac{2(a-x)}{a-y}$ when reduced to its lowest terms to be equal to p/q, so that p and q are prime to one another.

Then $2q(a-x)=p(a-y)$ and $p(a+x)=q(a+y)$.

Whence $(p^2+2q^2)a+(p^2-2q^2)x=2pqa$ or $x/a=\dfrac{p^2+2q^2-2pq}{2q^2-p^2}$.

The numerator being evidently positive, the denominator $(2q^2-p^2)$ must be also.

Since x/a is in its lowest terms, the fraction $\dfrac{p^2+2q^2-2pq}{2q^2-p^2}$, when reduced to its lowest terms, must have its numerator equal to x, and its denominator to a. But it is evident that p and q being prime to one another, the numerator and denominator can have no common factor, unless it be 2, in the event of p being even. Therefore if p be odd,

$$x=p^2+2q^2-2pq\ldots\ldots(1).$$

$$y=2q^2-p^2\ldots\ldots(2).$$

Whence by the equation $q(a+y)=p(a+x)$ we get

$$y=4pq-2q^2-p^2\ \ldots\ .(3).$$

If p be even,

$$x=\tfrac{1}{2}(p^2+2q^2-2pq)=2(\tfrac{1}{2}p)^2+q^2-2(\tfrac{1}{2}p)q,$$
$$a=\tfrac{1}{2}(2q^2-p^2)=q^2-2(\tfrac{1}{2}p)^2,$$
$$y=\tfrac{1}{2}(4pq-2q^2-p^2)=4(\tfrac{1}{2}p)q-q^2-2(\tfrac{1}{2}p)^2.$$

These equations are of the same form as the preceding, except in the case of a. For x and y we may take them in all cases,

$$\left.\begin{array}{l}x=p^2+2q^2-2pq\\ y=4pq-2q^2-p^2\end{array}\right\}\ldots\ldots(4).$$

Now from the equation $2y^2-x^2=b^2$, we have

$$2(y^2-b^2)=x^2-b^2.$$

Whence as before we have,

$$2s(y-b)=r(x-b),\quad r(y+b)=s(x+b),$$

r and s being prime to one another. Whence as before,

$$\left.\begin{array}{l} \left.\begin{array}{l} x=4rs-2s^2-r^2 \\ y=2s^2+r^2-2rs \end{array}\right\} \begin{array}{c} \text{or quantities of the same form} \\ \text{when } r \text{ is even,} \end{array} \\ b=2s^2-r^2,\ \text{or } =s^2-2(\tfrac{1}{2}r)^2,\ \text{when } r \text{ is even,} \end{array}\right\}\ldots\ldots(5).$$

Equating the two values of x and y we have $2pq=2rs$.

$\therefore\ x-pq=x-rs$ or $p^2+2q^2-3pq=3rs-2s^2-r^2$ or $(2q-p)(q-p)=(2s-r)(r-s)$.

Now from the equation $p(a+x)=q(a+y)$ we see that $q>p$, for $x>y$.
Then $q-p$ and $2q-p$ are both positive.
Similarly from the equation $r(y+b)=s(x+b)$, we see that $r>s$.
And since $(2s-r)(r-s)$ is positive, $2s-r$ is also positive.
$\therefore\ r>s<2s$. Now we may write

$$\left.\begin{array}{l} n(q-p)=m(r-s) \\ m(2q-p)=n(2s-r) \end{array}\right\}\ldots\ldots(6).$$

where m and n are positive integers, prime to one another.

$$\therefore\ (2m^2+n^2)q-(m^2+n^2)p=mns,$$

$$2(m^2+n^2)q-(m^2+2n^2)p=mnr.$$

$\therefore\ [(2m^2+n^2)q-(m^2+n^2)p][2(m^2+n^2)q-(m^2+2n^2)p]=m^2n^2rs=m^2n^2pq$, or
$2(m^2+n^2)(2m^2+n^2)q^2-2(2m^2+n^2)(m^2+2n^2)pq+(m^2+n^2)(2n^2+m^2p^2=0$.

Since the value of q/p derived from this equation must be rational,

$$(2m^2+n^2)^2(m^2+2n^2)-2(m^2+n^2)(2m^2+n^2)(m^2+2n^2)$$

must be a square, *i. e.* $(2m^2+n^2)(m^2+2n^2)m^2n^2$ must be a square, and $(2m^2+n^2)(m^2+2n^2)$ must be a square.

Therefore either $2m^2+n^2$ and m^2+2n^2 are both squares, or when divided by a common factor, the quotients are squares.

In the first case let $2m^2+n^2=h^2$, $m^2+2n^2=k^2$, then $m^2+n^2=\frac{1}{3}(h^2+k^2)$.

Now the sum of two squares is not divisible by 3, unless each square is divisible by 3, and therefore by 9. Hence $\frac{1}{9}(h^2+k^2)$ is an integer, whence $\frac{1}{3}(m^2+n^2)$ is an integer ; whence m and n are both divisible by 3, which is contrary to the hypothesis that m and n have no common factor. Therefore $2m^2+n^2$ and m^2+2n^2 have a common factor. This factor must be a factor also of their sum, *i. e.* $3(m^2+n^2)$.

But m^2+n^2 is prime to $2m^2+n^2$, since if they have a common factor it is also a factor of m^2+n^2 and $(2m^2+n^2)-(m^2+n^2)$, or of m^2+n^2 and m^2, or of n^2 and m^2, which is again contrary to hypothesis.

Therefore the common factor of $2m^2+n^2$ and $m^2+2n^2=3$, and we may put

$$\left.\begin{array}{l} 2m^2+n^2=3h^2 \\ m^2+2n^2=3k^2 \end{array}\right\}\ldots\ldots(7).$$

From the first of these, $2(m^2-h^2)=h^2-n^2$, me may now take $\dfrac{h-n}{2(m-h)}$ in its lowest terms$=\lambda/\mu$, so that

$$2\lambda(m-h)=\mu(h-n),\quad \mu(m+h)=\lambda(h+n),$$

whence $(2\lambda^2+\mu^2)m-(2\lambda^2-\mu^2)h=2\lambda\mu h$, and $m/h=\dfrac{2\lambda\mu+2\lambda^2-\mu^2}{2\lambda^2+\mu^2}$.

Whence as before,

$$\left.\begin{array}{l} h=2\lambda^2+\mu^2 \\ m=2\lambda\mu+2\lambda^2-\mu^2 \\ n=2\lambda\mu-2\lambda^2+\mu^2 \end{array}\right\}\ldots\ldots(8),$$

or the halves of these, respectively, which will have the same form, except that m and n are transposed.

Similarly from the equation $m^2+2n^2=3k^2$, we get

$$m^2-k^2=2(k^2-n^2) \text{ and } 2\lambda_1(k-n)=\mu_1(m-k).$$

Whence
$$\left.\begin{array}{l} k=2\lambda_1^2+\mu_1^2 \\ m=2\lambda_1\mu_1-2\lambda_1^2+\mu_1^2 \\ n=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2 \end{array}\right\}\ldots\ldots(9),$$

or the halves of these, respectively, which will have the same form except that the expressions for m and n will be interchanged.

From the first set $m+n=4\lambda_1\mu_1$, and from the second $m+n=4\lambda_1\mu_1$.

$\therefore\ \lambda\mu=\lambda_1\mu_1$.

Also $2\lambda\mu+2\lambda^2-\mu^2=2\lambda_1\mu_1-2\lambda_1^2+\mu_1^2$,

or $=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2$.

In the first case, $2\lambda^2-\mu^2=-2\lambda_1^2+\mu_1^2$.

$\therefore\ 2(\lambda^2+\lambda_1^2)=\mu^2+\mu_1^2$,

$2\lambda_1^2(\lambda^2+\lambda_1^2)=\lambda_1^2\mu^2+\lambda_1^2\mu_1^2=\lambda_1^2\mu^2+\lambda^2\mu^2=(\lambda^2+\lambda_1^2)\mu^2$.

$\lambda^2+\lambda_1^2$ cannot be zero.

∴ $2\lambda_1^2=\mu^2$, or $\sqrt{2}\lambda_1=\mu$, which is impossible since λ_1 and μ are integers. We must now therefore try the second case,

$$2\lambda\mu+2\lambda^2-\mu^2=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2.$$

Whence $2(\lambda^2-\lambda_1^2)=\mu^2-\mu_1^2$,
$$2\lambda_1^2(\lambda^2-\lambda_1^2)=\lambda_1^2\mu^2-\lambda_1^2\mu_1^2=\lambda_1^2\mu^2-\lambda^2\mu^2=(\lambda_1^2-\lambda^2)\mu^2.$$
∴ $(2\lambda_1^2+\mu^2)(\lambda^2-\lambda_1^2)=0$.
$2\lambda_1^2+\mu^2$ cannot be zero.
∴ the only solution is $\lambda^2-\lambda_1^2=0$, or $\lambda=\lambda_1$.
Hence from $\lambda\mu=\lambda_1\mu_1$, $\mu=\mu_1$.
Now $h=2\lambda^2+\mu^2=2\lambda_1^2+\mu_1^2=k$ [from (8) and (9)].
∴ $2m^2+n^2=m^2+2n^2$ [from (7)].
∴ $m^2=n^2$.
But m and n have by hypothesis no common factor (greater than unity).
∴ $m=n=1$.
∴ $q-p=r-s$ [from (6)], and $pq=rs$.
∴ $q+p=r+s$, and $q=r$, $p=s$. Also $2q-p=2s-r=2p-q$.
∴ $q=p$, and as they have no common measure, each$=1$.
∴ $p=q=r=s=1$, and

$$\left.\begin{aligned}x&=p^2+2q^2-2pq=1\\ y&=4pq-2q^2-p^2=1\end{aligned}\right\}\text{from (4).}$$

$$\begin{aligned}a&=\sqrt{(2x^2-y^2)}=1\\ b&=\sqrt{(2y^2-x^2)}=1.\end{aligned}$$

(These values of x and y are also derivable from the equations

$$\begin{aligned}x&=4rs-2s^2-r^2\\ y&=2s^2+r^2-2rs,\end{aligned}$$

so that all the equations are consistent.)

∴ $x=y=a=b=1$ is the only solution in integers of the two equations when x and y are prime to one another.

Of course an infinite number of solutions may be obtained by multiplying x, y, a, and b by a common factor, C, so that

$$2C^2-C^2=C^2 \text{ for } 2x^2-y^2=a^2,$$
$$2C^2-C^2=C^2 \text{ for } 2y^2-x^2=b^2.$$